

# A spherically symmetric dust-space-time with a NUT-like rotation

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## 1 Introduction

In this paper we present a deduction of a solution (already discovered in 1983 by Lukács *et al*) of the Einstein's equations, employing the Mitskievich's field theoretic description of perfect fluids. This solution describes a dust-space-time with a spherical-like symmetry and a NUT-like rotation. This solution is of Petrov type  $D$ , and has an isometry group  $G_4$ . It also admits closed timelike geodesics. It has Minkowski space-time as a limit, when both dust and rotation disappear.

This paper is organized as follows, in Section two the Mitskievich's field theoretic description of perfect fluids is presented, then in Section three we show this solution and its properties.

Below we are working in four space-time dimensions with signature  $(+, -, -, -)$ , Greek indices being four-dimensional. The Ricci tensor is  $R_{\mu\nu} = R^\alpha_{\mu\nu\alpha}$ , thus Einstein's equations take the form  $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\kappa T_{\mu\nu}$ .

## 2 Rotating fluids in field-theoretic-description

Perfect fluids can be conveniently described with use of the Lagrangian formalism, especially in the absence of rotation [2, 3]. In this case they are represented via the 2-form field potential  $B = \frac{1}{2!}B_{\mu\nu}dx^\mu \wedge dx^\nu$ , the respective field intensity being  $G = dB = \frac{1}{2}B_{\mu\nu;\lambda}dx^\lambda \wedge dx^\mu \wedge dx^\nu$  ( $B_{[\mu\nu;\lambda]} \equiv B_{[\mu\nu,\lambda]}$ ,

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and  $G_{\lambda\mu\nu} = B_{\lambda\mu,\nu} + B_{\mu\nu,\lambda} + B_{\nu\lambda,\mu}$ ) whose invariant  $J = *(G \wedge *G)$  (we shall also denote  $*G = \tilde{G}$ ) is used in constructing the fluid Lagrangian density  $\mathcal{L} = \sqrt{-g}L(J)$ . Here the Hodge star  $*$  denotes, as usual, a generalization of the dual conjugation applied to Cartan exterior forms: with an  $r$ -form  $\alpha = \alpha_{\nu_1 \dots \nu_r} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_r}$ , it yields a  $(4-r)$ -form  $*\alpha$  with the components  $(*\alpha)_{\nu_1 \dots \nu_{4-r}} = \frac{1}{r!} E_{\nu_1 \dots \nu_{4-r} \nu_{5-r} \dots \nu_4} \alpha^{\nu_{5-r} \dots \nu_4}$  where  $E_{\lambda\mu\nu} = \sqrt{-g} \epsilon_{\lambda\mu\nu}$  and  $E^{\lambda\mu\nu} = -(1/\sqrt{-g}) \epsilon^{\lambda\mu\nu}$  are co- and contravariant axial skew rank-4 tensors,  $\epsilon_{\lambda\mu\nu} = \epsilon_{[\lambda\mu\nu]}$ ,  $\epsilon_{0123} = +1$  being the Levi-Civita symbol (*cf.* somewhat other notations in [1]).

The reason why this description of perfect fluids is valid, is simply the fact that the stress-energy tensor of a 2-form field is

$$T_\alpha^\beta = 2J \frac{dL}{dJ} b_\alpha^\beta - L \delta_\alpha^\beta \quad (1)$$

where

$$b_\alpha^\beta = \delta_\alpha^\beta - u_\alpha u^\beta, b_\alpha^\beta u^\alpha = 0 = b_\alpha^\beta u_\beta, u = J^{-1/2} \tilde{G}. \quad (2)$$

When  $u$  is timelike ( $u \cdot u = +1$ , as we above supposed it to be), we come to the usual perfect fluid whose (arbitrary) equation of state is determined by the dependence of  $L$  on its only argument,  $J$  (see [2], [3]) [however when it is spacelike, the ‘fluid’ is tachyonic (see for some details [4], subsection 3.2)]. Since  $b_\alpha^\beta$  is the projector on the (local) subspace orthogonal to the congruence of  $u$ , the latter is an eigenvector of the stress-energy tensor with the eigenvalue  $(-L)$ ,  $T_\alpha^\beta u^\alpha = -L u^\beta$ , while any vector orthogonal to  $u$  is also eigenvector, now with the three-fold eigenvalue  $2J \frac{dL}{dJ} - L$ . This is the property of the stress-energy tensor of a perfect fluid possessing the proper mass density  $\mu$  and pressure  $p$  (in its local rest frame):

$$\mu = -L, \quad p = L - 2J \frac{dL}{dJ}. \quad (3)$$

Below we consider perfect fluids characterized by the simplest equation of state

$$p = (2k - 1)\mu \quad (4)$$

(the frequently used notation is  $2k = \gamma$ ) which correspond to the Lagrangian  $L = -\sigma |J|^k$ ,  $\sigma > 0$ . In a four-dimensional spacetime, the important special cases are: the incoherent dust ( $p = 0$ ) for  $k = 1/2$ , intrinsically relativistic incoherent radiation ( $p = \mu/3$ ) for  $k = 2/3$ , and hyperrelativistic stiff matter ( $p = \mu$ ) for  $k = 1$ .

However the 2-form field equation which follows from the above Lagrangian,

$$\left( \sqrt{-g} \frac{dL}{dJ} G^{\lambda\mu\nu} \right)_{,\nu} = 0 \iff d \left( J^{1/2} \frac{dL}{dJ} u \right) = 0, \quad (5)$$

only means that the  $\tilde{G}$  (equivalently,  $u$ ) congruence is non-rotating. To describe a rotating fluid, one has to introduce in (5) a non-zero right-hand side. This, in a sharp contrast to the usual equations of mathematical physics (*cf.*, for example, electrodynamics), *cannot then be interpreted as a usual source term* (this was stressed in [4]): its meaning essentially is to indicate the presence of rotation ( $u \wedge du \neq 0$ ). To this end it is necessary to consider one more field which we call the Machian one, a 3-form field  $C$  with the intensity  $W = dC$  (see [2, 3]). In terms of  $L(K)$ ,  $K = -(1/4!) W_{\kappa\lambda\mu\nu} W^{\kappa\lambda\mu\nu} = \tilde{W}^2$ , its equations reduce to

$$\left( \sqrt{-g} \frac{dL}{dK} W^{\kappa\lambda\mu\nu} \right)_{,\nu} = 0 \Rightarrow K^{1/2} \frac{dL}{dK} = \text{const.} \quad (6)$$

We use also the duality relations  $B^{\mu\nu} = \frac{1}{2} E^{\mu\nu\alpha\beta} B_{\alpha\beta}$ ,  $G_{\lambda\mu\nu} = \tilde{G}^\kappa E_{\kappa\lambda\mu\nu}$ ,  $W_{\kappa\lambda\mu\nu} = \tilde{W} E_{\kappa\lambda\mu\nu}$ . Moreover,  $B^{\mu\nu}_{*\nu} \equiv -( *G)^\mu$ .

Since we were confronted with the no rotation property of perfect fluid when the rank 2 field was considered to be free, the only remedy now is to introduce a non-trivial “source” term in the  $r = 2$  field equations, thus to consider the non-free field case or, at least, to include in the Lagrangian a dependence on the rank 2 field potential  $B$ . The simplest way to do this is to introduce in the Lagrangian density dependence on a new invariant  $J_1 = -B_{[\kappa\lambda} B_{\mu\nu]} B^{[\kappa\lambda} B^{\mu\nu]}$  which does not spoil the structure of stress-energy tensor, simultaneously yielding a “source” term (thus permitting to destroy the no rotation property) without changing the divergence term in the  $r = 2$  field equations. We shall use below three invariants: the obvious ones,  $J$  and  $K$ , and the just introduced invariant of the  $r = 2$  field *potential*,  $J_1$ . Then

$$B_{[\kappa\lambda} B_{\mu\nu]} = -\frac{2}{4!} B_{\alpha\beta} B^{*\alpha\beta} E_{\kappa\lambda\mu\nu}. \quad (7)$$

Thus  $J_1^{1/2} = 6^{-1/2} B_{\alpha\beta} B^{*\alpha\beta}$ . In fact,  $J_1 = 0$ , if  $B$  is a simple bivector ( $B = a \wedge b$ ,  $a$  and  $b$  being 1-forms). This *cannot however annul* the expression which this invariant contributes to the  $r = 2$  field equations: up to a factor,

it is equal to  $\partial J_1^{1/2}/\partial B_{\mu\nu} \neq 0$ . Thus let the Lagrangian density be

$$\mathcal{L} = \sqrt{-g} \left( L(J) + M(K) J_1^{1/2} \right), \quad (8)$$

so that the  $r = 2$  field equations take the form (*cf.* (5))

$$d \left( \frac{dL}{dJ} \tilde{G} \right) = \sqrt{\frac{2}{3}} M(K) B \Leftrightarrow \left( \sqrt{-g} \frac{dL}{dJ} G^{\alpha\beta\nu} \right)_{,\nu} = \sqrt{-g} \sqrt{\frac{2}{3}} M(K) B^{\alpha\beta}{}_{*}. \quad (9)$$

In their turn, the  $r = 3$  field equations (*cf.* (6)) yield the first integral

$$J_1^{1/2} K^{1/2} \frac{dM}{dK} = \text{const} \equiv 0 \quad (10)$$

(since  $J_1 = 0$ ). We know from [1, 2] that  $K$  (hence,  $M$ ) *arbitrarily* depends on the space-time coordinates, if only the  $r = 3$  field equations are taken into account, and the Machian field  $K$  has to be essentially non-constant.

The stress-energy tensor which corresponds to the new Lagrangian density (8), automatically coincides with its previous form (1), since  $J_1 = 0$ . For a perfect fluid with the equation of state  $p = (2k - 1)\mu$ , one finds  $L = -\sigma J^k$ , thus  $T_{\alpha}^{\beta} = -2k L u_{\alpha} u^{\beta} + (2k - 1) L \delta_{\alpha}^{\beta}$ . Then the traditional perfect fluid language is obviously related with that of the  $r = 2$  and  $r = 3$  fields:

$$\left. \begin{aligned} \mu = -L = \sigma J^k, \quad \tilde{G}^{\mu} = \Xi \delta_t^{\mu}, \quad \Xi = \frac{1}{\sqrt{g_{00}}} \left( \frac{\mu}{\sigma} \right)^{1/(2k)}, \\ G = dB = d \left( \frac{\sqrt{3/2}}{M(K)} \right) \wedge d \left( \frac{dL}{dJ} \tilde{G} \right) \end{aligned} \right\} \quad (11)$$

(*cf.* (9)). The function  $M$  depends arbitrarily on coordinates; thus one can choose its adequate form using the last relation without coming into contradiction with the dynamical Einstein–Euler equations.

When one describes a fluid in its proper basis,  $u = J^{-1/2} \tilde{G} = \theta^{(0)}$ , the rotation of the fluid's co-moving reference frame is defined as  $\omega = * \left( \theta^{(0)} \wedge d\theta^{(0)} \right) = J^{-1} * (\tilde{G} \wedge d\tilde{G})$ . Let us assume  $\theta^{(0)} = e^{\alpha}(dt + f d\phi)$  where  $\alpha$  and  $f$  are functions of coordinates (usually determined *via* Einstein's equations), *cf.* the examples of metrics considered in the next Sections, though in these examples are treated Einstein–Maxwell fields and still not the perfect fluid solutions. It is inevitable to conclude that the field theoretic approach to perfect fluids automatically gives hints and even concrete relations (often

having a simple algebraic form) imposed upon these and other functions characterizing the metric tensor and the 2-form field, as well as the Machian one. This makes it possible to substantially simplify the treatment of Einstein's equations.

### 3 A spherically symmetric dust-space-time with NUT-like rotation

We begin considering the following line element

$$ds^2 = e^{2\alpha(r)}[dt + l \cos \vartheta d\varphi]^2 - e^{2\beta(r)}dr^2 - r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2),$$

together with the orthonormal basis

$$\theta^{(0)} = e^\alpha[dt + l \cos \vartheta d\varphi], \quad \theta^{(1)} = e^\beta dr, \quad \theta^{(2)} = r d\vartheta, \quad \theta^{(3)} = r \sin \vartheta d\varphi.$$

Now we introduce the following 2-form, in order to satisfy (9), the 2-form should have the structure,

$$B = F(r) \sin \vartheta d\vartheta \wedge d\varphi,$$

to avoid a dependence of  $\vartheta$  in  $J$ , we have introduced the The respective field intensity being

$$G = dB = \frac{F'(r)e^{-\beta}}{r^2} \theta^{(1)} \wedge \theta^{(2)} \wedge \theta^{(3)},$$

and

$$\tilde{G} = *G = \frac{F'(r)e^{-\beta}}{r^2} \theta^{(0)}.$$

The corresponding invariant

$$J = \tilde{G} \cdot \tilde{G} = \left( \frac{F'}{r^2} e^{-\beta} \right)^2.$$

When we consider the dust case,  $L = -\sigma\sqrt{J}$ , the field equations (9) take the form

$$d \left[ -\frac{\sigma}{2} e^\alpha (dt + l \cos \vartheta d\varphi) \right] = \sqrt{\frac{2}{3}} M(K) F(r) \sin \vartheta d\vartheta \wedge d\varphi.$$

From them we conclude that  $\alpha = 0$  and

$$M(K) = \sqrt{\frac{3}{8}} \frac{l}{\sigma F'}.$$

With this conclusion Einstein's equations are

$$G_{(0)(0)} = \frac{1}{r^2} \left( e^{-2\beta} - 1 - \frac{l^2}{4r^2} \right) + \frac{1}{r} \left( e^{-2\beta} \right)' - \frac{l^2}{2r^4} = -\varkappa\mu$$

$$G_{(1)(1)} = -\frac{1}{r^2} \left( e^{-2\beta} - 1 - \frac{l^2}{4r^2} \right) = 0,$$

$$G_{(2)(2)} = G_{(3)(3)} = -\frac{1}{2r} \left( e^{-2\beta} \right)' - \frac{l^2}{4r^4} = 0.$$

Immediately we find that

$$e^{-2\beta} = 1 + \frac{l^2}{4r^2}, \quad \mu = \frac{l^2}{\varkappa r^4}.$$

From the field theoretic description of the mass density

$$\mu = \sigma \sqrt{J} = \frac{\sigma \sqrt{4r^2 + l^2}}{2r^3} F',$$

we find

$$F(r) = -\frac{2l}{\sigma \varkappa} \ln \left( \frac{l + \sqrt{4r^2 + l^2}}{2r} \right)$$

and from (3)

$$M(K) = -\sqrt{\frac{3}{32}} \varkappa \left[ \ln \left( \frac{l + \sqrt{4r^2 + l^2}}{2r} \right) \right]^{-1}.$$

### 3.0.1 The Solution

Doing the change  $l \rightarrow 2l$ , the solution becomes

$$ds^2 = [dt + 2l \cos \vartheta d\varphi]^2 - \frac{r^2}{r^2 + l^2} dr^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (12)$$

with

$$\mu = \frac{4l^2}{\varkappa r^4}, \quad B = -\frac{4l}{\sigma \varkappa} \ln \left( \frac{l + \sqrt{r^2 + l^2}}{r} \right) \sin \vartheta d\vartheta \wedge d\varphi$$

and

$$M(K) = -\sqrt{\frac{3}{32}} \left[ \ln \left( \frac{l + \sqrt{r^2 + l^2}}{r} \right) \right]^{-1}.$$

Note that if  $l = 0$ , we arrive to the Minkowski space-time.

### 3.0.2 Petrov type and isometries

This solution is of Petrov type  $D$ , and it possesses an isometry group  $G_4$ . Killing vectors are

$$\begin{aligned} \xi_{[0]} &= \partial_t, \\ \xi_{[1]} &= \partial_\varphi, \\ \xi_{[2]} &= 2l \frac{\cos \varphi}{\sin \vartheta} \partial_t - \sin \varphi \partial_\vartheta - \cot \vartheta \cos \varphi d\varphi, \\ \xi_{[3]} &= -2l \frac{\sin \varphi}{\sin \vartheta} \partial_t - \cos \varphi \partial_\vartheta - \cot \vartheta \sin \varphi d\varphi. \end{aligned}$$

They satisfy

$$[\xi_{[0]}, \xi_{[i]}] = 0, \quad [\xi_{[i]}, \xi_{[j]}] = \varepsilon_{ijk} \xi_{[k]},$$

where  $i, j, k = 1, 2, 3$ . We see the space-time has a spherical-like symmetry.

### 3.0.3 Closed timelike geodesics

The angular coordinate  $\varphi$  plays the rôle of a timelike coordinate for some values of the other coordinates. Now consider a motion with constant  $t$ ,  $r$  and  $\vartheta$ .

From geodesic equation we find the first integral

$$(4l^2 \cos^2 \vartheta - r^2 \sin^2 \vartheta) \left( \frac{d\varphi}{ds} \right) = \Lambda.$$

Substituting it in the line element  $ds^2$ , we arrive to

$$\frac{\Lambda^2}{4l^2 \cos^2 \vartheta - r^2 \sin^2 \vartheta} = 1.$$

Thus, if  $4l^2 \cos^2 \vartheta - r^2 \sin^2 \vartheta > 0$ , we find that our solution accepts closed timelike geodesics for

$$\Lambda = \sqrt{4l^2 \cos^2 \vartheta - r^2 \sin^2 \vartheta}.$$

In the particular case  $r = 2l$ , it is required that  $\vartheta < \pi/4$  ó  $\vartheta > 3\pi/4$ .

### 3.0.4 The co-moving reference frame

The co-moving reference frame with dust-particles is described by the monad (dust 4-velocity, see [1]).

$$\tau = \theta^{(0)} = dt + 2l \cos \vartheta d\varphi.$$

This reference frame is rotating

$$\omega := \frac{1}{2} * (\tau \wedge d\tau) = -\frac{l}{r\sqrt{l^2 + r^2}} dr,$$

but has no acceleration, expansion, and shear (the rate of strain tensor vanishes).

### 3.0.5 Curvature

We see that the curvature presents a strong singularity

$$R_{(0)(2)(0)(2)} = R_{(0)(3)(0)(3)} = R_{(1)(2)(1)(2)} = R_{(1)(3)(1)(3)} = -\frac{l^2}{r^4}$$

$$R_{(2)(3)(2)(3)} = -2\frac{l^2}{r^4}.$$

$$R_{(0)(1)(2)(3)} = 2R_{(1)(2)(3)(0)} = -2R_{(1)(3)(2)(0)} = 2\frac{l\sqrt{l^2 + r^2}}{r^4}$$

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